



UNIVERSITY OF
LIVERPOOL

JANUARY EXAMINATIONS 2011

Bachelor of Science: Year 3
Master of Physics: Year 3
Master of Physics: Year 4

STATISTICAL AND LOW TEMPERATURE PHYSICS

TIME ALLOWED: 3 hours

INSTRUCTIONS TO CANDIDATES

Answer **all** questions.

Question 1 carries 50% of the total marks.

Questions 2 and 3 each carry 25% of the total marks.

Answer either part (a) or part (b) of questions 2 and 3.

In the event of a student answering both parts of an either/or question and not clearly crossing out one answer, only the answer to part (a) of the question will be marked.

The marks allotted to each part of a question are indicated in square brackets.

All symbols have their usual meanings unless otherwise stated.

Question 1.

(a) A salt in magnetic field contains N spin $\frac{1}{2}$ ions at temperature T . The magnetic energy levels for each of the ions are $-\epsilon$ and ϵ .

i) Write an expression for the population of each level in terms of N , ϵ , and T . [2]

ii) Show that the internal energy can be written as

$$U = -N\epsilon \frac{\exp(\epsilon/k_B T) - \exp(-\epsilon/k_B T)}{\exp(\epsilon/k_B T) + \exp(-\epsilon/k_B T)}. \quad [2]$$

iii) Find the low and high T limits of U . [2]

iv) Sketch the variation of U , and heat capacity C , with T . [2]

(b) A 1 m^3 box contains one mole of helium at room temperature.

i) Find the average kinetic energy of the atoms. [2]

ii) Find the corresponding wavevector. [2]

iii) The total number of states with wavevector below k is $G(k) = 4Vk^3/3\pi^2$, where V is the volume. Find the number of states below the average kinetic energy of the atoms. [2]

iv) Estimate the probability that one of these states is occupied. [2]

v) How does this affect the most probable macrostate? [2]

(c) A system of particles, at temperature T , occupies a set of energy states. The probability that a state at energy ϵ is occupied, is $f(\epsilon)$.

i) Write down the expressions for $f(\epsilon)$ if the particles are fermions, and if they are bosons. [2]

ii) Why are they different? [2]

iii) Sketch a diagram to show how the energy levels of fermions are occupied near 0 K. [2]

iv) Sketch a diagram to show how the energy levels of bosons are occupied near 0 K. [2]

(d)

- i) The presence of a magnetic field in a macroscopic wavefunction of electrons must produce a current. How does this explain the Meissner's effect? [2]
- ii) Discuss how this gives rise to London's penetration depth. [3]
- iii) Sketch the heat capacity versus temperature graph for a superconductor, above and below the transition temperature. How does this suggest the existence of an energy gap? [3]

(e) A body can move with zero resistance through superfluid ^4He .

- i) What excitations are possible in superfluid ^4He ? [2]
- ii) Consider an excitation of energy E and momentum p . The body's velocity must be above E/p before excitation is possible. What are the conservation laws that lead to this? [2]
- iii) Sketch the dispersion relation of the excitations in superfluid ^4He . Draw the line with gradient equal to the minimum E/p . [2]
- iv) Why does the body experience no resistance when its velocity is below the minimum E/p ? [2]

(f)

- i) State the electronic, nuclear and total angular momenta of ^3He and ^4He atoms. [2]
- ii) Explain why one is a fermion and the other is a boson. [2]
- iii) The atoms in liquid ^3He at 2 K occupy the energy levels in a certain way. Sketch a picture to show this. [2]
- iv) The atoms in liquid ^4He at 2 K occupy the energy levels in a certain way. Sketch a picture to show this. [2]

Question 2. Answer either (a) or (b)

(a) One mole of copper is at 1 K. Each atom supplies two conduction electrons.

- i) Assuming that the electrons behave like an ideal gas, write down the expression for the average kinetic energy of the electrons. Hence find the heat capacity. [3]
- ii) At 1 K, the measured heat capacity is 0.6 mJ/K. Explain why it is different. [2]
- iii) With the help of a graph, estimate the energy range of the electrons that are excited above the Fermi energy at temperature T. [4]
- iv) Using the density of states for the ideal gas,

$$g(\epsilon) = \frac{4m\pi V}{h^3} \sqrt{2m\epsilon} ,$$

derive an expression for the number, n, of excited electrons. (Molar volume of copper is 7.11 cm³).

- [4]
- v) Why is it reasonable to suppose that these electrons behave like the ideal gas? [4]
- vi) Derive an expression for the heat capacity using the ideal gas assumption. [4]
- vii) Find the Fermi energy. Calculate the heat capacity for copper at 1 K. Compare with the measured value and comment. [4]

(b) N conduction electrons move freely inside a cube of metal of side L .

i) Write quantised values for the wavevector components k_x, k_y, k_z . [2]

ii) Illustrate the allowed states in k space. [2]

iii) Show that the number of states with wavevectors in the range k to $k+dk$ is

$$g(k)dk = \frac{2Vk^2 dk}{\pi^2},$$

where V is the volume L^3 . [6]

iv) Write the relation between k and energy ϵ . [1]

v) Show that the number of states with energy in the range ϵ to $\epsilon+d\epsilon$ is

$$g(\epsilon) = \frac{2mV}{\pi^2 \hbar^3} \sqrt{2m\epsilon}. \quad [3]$$

vi) The probability that a state with energy ϵ will be occupied is $f(\epsilon)$. Sketch graphs of $f(\epsilon)$ versus ϵ for electrons for temperature $T = 0$ K, and for nonzero T much less than the Fermi temperature T_F .

Indicate the Fermi energy E_F . [3]

vii) Show that E_F is given by

$$E_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}. \quad [5]$$

viii) Silver has a molar volume of $10.27 \times 10^{-6} \text{ m}^3$. Each atom contributes one conduction electron.

Find the Fermi energy. [3]

Question 3. Answer either (a) or (b)

(a)

i) Why is the Bose Einstein condensate a good candidate for explaining superfluidity and superconductivity? [5]

ii) Using a sketch of the ~~Fermi Dirac~~ ^{Bose Einstein} distribution graph, explain what happens to the chemical potential μ of a boson gas as temperature falls to 0 K? [5]

iii) In terms of the density of states $g(\epsilon)$, write down the expression for the number of particles N . Explain when and why the number of excited bosons may be written as

$$N_{ex} = \int_0^{\infty} \frac{g(\epsilon)d\epsilon}{\exp(\epsilon/k_B T) - 1}. \quad [5]$$

iv) Given that the solution is

$$N_{ex} = \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} 2.612V,$$

explain how to find the condensation temperature T_{BE} . Find the value for liquid ^4He , with molar volume ~~36.84~~ ^{27.58} cm^3 . [5]

v) In terms of $g(\epsilon)$, write down the expression for the energy U below T_{BE} . Given that

$$U = 0.7704 k_B N \frac{T^{5/2}}{T_{BE}^{3/2}}$$

is the solution, derive the heat capacity C . Calculate the value of C for one mole of ^4He at T_{BE} , and sketch the graph of C versus T . [5]

(b)

- i) Describe qualitatively the basic features of the theory of superconductivity. [5]
- ii) Explain qualitatively what happens as the temperature of a superconductor rises above the critical temperature. [2]
- iii) Explain qualitatively what happens as a magnetic field higher than the critical field is applied to a superconductor. [2]
- iv) Describe the isotope effect in superconductivity. What does it prove? [4]
- v) Describe the Meissner effect. [5]
- vi) Give two main characteristics of high- T_C superconductor materials. [2]
- vii) How does the magnetic flux quantum confirm the BCS theory? Mention one important application of magnetic flux quantization. [5]

CONSTANTS

Speed of light in vacuum	c	$=$	$3.00 \times 10^8 \text{ ms}^{-1}$
Permeability of vacuum	μ_0	$=$	$4\pi \times 10^{-7} \text{ Hm}^{-1}$
		$=$	$4\pi \times 10^{-7} \text{ VsA}^{-1}\text{m}^{-1}$
Permittivity of vacuum	ϵ_0	$=$	$8.85 \times 10^{-12} \text{ Fm}^{-1}$
		$=$	$8.85 \times 10^{-12} \text{ AsV}^{-1}\text{m}^{-1}$
Elementary charge	e	$=$	$1.60 \times 10^{-19} \text{ C}$
Planck constant	h	$=$	$6.63 \times 10^{-34} \text{ Js}$
	$h/2\pi = \hbar$	$=$	$1.05 \times 10^{-34} \text{ Js}$
Avogadro constant	N_A	$=$	$6.02 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	k_B	$=$	$1.38 \times 10^{-23} \text{ JK}^{-1}$
Gas constant	R	$=$	$8.31 \text{ JK}^{-1}\text{mol}^{-1}$
Unified atomic mass constant	m_u	$=$	$1.66 \times 10^{-27} \text{ kg}$
		$=$	931.5 MeVc^{-2}
Electron mass	m_e	$=$	$9.11 \times 10^{-31} \text{ kg}$
Proton mass	m_p	$=$	$1.67 \times 10^{-27} \text{ kg}$
Gravitational constant	G	$=$	$6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}$
Acceleration due to gravity	g	$=$	9.8 ms^{-2}
Bohr magneton	μ_B	$=$	$9.27 \times 10^{-24} \text{ JT}^{-1}$